Math 113: Exam 2
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Name: KEY

You are not permitted to use calculators, textbooks, friends, tutors, etc. for this test. You may have a one-page, one-sided sheet of paper (normal size) with notes during the exam, and you must turn this sheet in with your test. You must show all your work for credit. Partial credit will be given when appropriate, but a correct answer does not guarantee full credit if work is not shown.

1. Let $f(x) = 2x^5 - 3x^4 + 3$. Calculate $f'(1)$.

$f'(x) = 10x^4 - 12x^3$

$f'(1) = 10 - 12 = -2$

2. Let $f(x) = 3\sin^5 x$. Calculate $f'(x)$.

Chain Rule

$$f'(x) = 3 \cdot 5\sin^4 x \cdot \frac{d}{dx}\sin x$$

$$= 15\sin^4 x \cos x$$

3. Find $f'(x)$ given

$$f(x) = \frac{x^{\frac{2}{3}}}{\cos(5x)}.$$

$$f'(x) = \frac{\frac{d}{dx}(x^{2/3})\cos(5x) - \frac{d}{dx}(\cos 5x) \cdot x^{2/3}}{\cos^2(5x)}$$

$$= \frac{\frac{2}{3}x^{-1/3}\cos(5x) - 5(-\sin(5x))\, x^{2/3}}{\cos^2(5x)}$$

$$= \frac{\frac{2}{3}x^{-1/3}\cos(5x) + 5x^{2/3}\sin(5x)}{\cos^2(5x)}.$$

4. Let $f(t) = (2t+1)^5(t^{-2}-5)^4$. Find $f'(t)$. You do not need to simplify your answer.

$$\frac{d}{dt}(2t+1)^5 \cdot (t^{-2}-5)^4 + \frac{d}{dt}(t^{-2}-5)^4 \cdot (2t+1)^5$$

$$= 5(2t+1)^4 \cdot 2 \cdot (t^{-2}-5)^4 + 4(t^{-2}-5)^3 \cdot \frac{d}{dt}(t^{-2}-5) \cdot (2t+1)^5$$

$$= 10(2t+1)^4(t^{-2}-5)^4 - 8(t^{-2}-5)^3 t^{-3} (2t+1)^5$$

5. Let $f(x) = \arcsin \frac{1}{x^2}$. Find $f'(x)$.

Let $u = \frac{1}{x^2}$. $f(x) = \arcsin u$

$= x^{-2}$ $\qquad \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$ $\qquad \frac{df}{du} = \frac{1}{\sqrt{1-u^2}}$

$\frac{du}{dx} = -2x^{-3}$ $\qquad \frac{df}{dx} = \frac{1}{\sqrt{1-(\frac{1}{x^2})^2}} \cdot -2x^{-3}$

$$= \frac{1}{\sqrt{1-\frac{1}{x^4}}} \cdot -2x^{-3} \cdot \frac{x^2}{x^2}$$

$$= \frac{-2x^{-1}}{x^2\sqrt{1-\frac{1}{x^4}}} = \frac{-2x^{-1}}{\sqrt{x^4-1}}$$

6. Suppose an ant is moving along a straight line according to the function

$$s(t) = t^3 - \frac{9}{2}t^2 + 6t,$$

where t is time and s is the position along the line at time t. Find the ant's acceleration each time the velocity is zero.

$v(t) = 3t^2 - 9t + 6$

$v(t) = 0$

$\iff 3t^2 - 9t + 6 = 0$

$\iff t^2 - 3t + 2 = 0$

$(t-2)(t-1) = 0$

$\Rightarrow t = 1, 2$

$a(t) = 6t - 9$

$a(1) = 6 - 9 = -3$

$a(2) = 12 - 9 = 3.$

7. Find an equation of the tangent line to the curve $x^2y^2 = (y+1)^2$ at the point $(2, 1)$.

$$\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(y+1)^2$$

$$\frac{d}{dx}(x^2y^2) = 2xy^2 + x^2\frac{d}{dx}(y^2)$$ product rule

$$= 2xy^2 + x^2 \cdot 2y\,y'$$

$$\frac{d}{dx}(y+1)^2 = 2(y+1)\,y'$$

$$2xy^2 + 2x^2y\,y' = 2(y+1)y'$$

$$2xy^2 = 2(y+1)y' - 2x^2y\,y'$$

$$= \left(2(y+1) - 2x^2y\right)y'$$

$$y' = \frac{2xy^2}{2(y+1) - 2x^2y}$$

At $(2,1)$ $\quad y' = \dfrac{2(2)(1^2)}{2(1+1) - 2(2^2)\cdot 1} = \dfrac{4}{4-8} = -1$

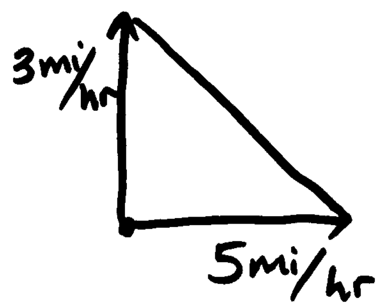
$y = -x + b$

$1 = -2 + b \Rightarrow b = 3$

$\boxed{y = -x + 3}$

8. There is a tree in the middle of a large field in which two kids are playing the following game. They have three long ropes. Each kid ties one end of a rope around a tree, and holds the other end. They share the third rope, each kid holding one end of the rope in his hands. When the game begins, the first child runs North at 3 mi/hr, and the second child runs East at 5 mi/hr. As they run, they hold onto the ends of the rope. When the first kid is 6 miles away from the tree, how fast is the area enclosed by the rope changing?

A = area of enclosed region



Let x = distance of kid going east from tree.

y = " " " " north " "

$A = \frac{1}{2}xy$ $\quad \frac{dx}{dt} = 5$ mi/hr

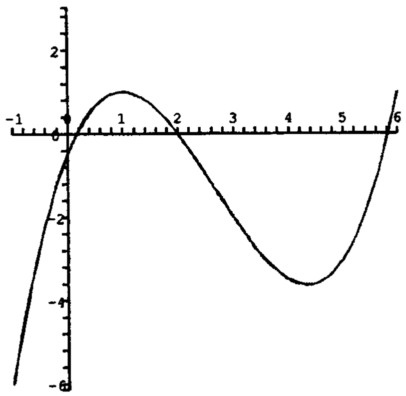
$\frac{dy}{dt} = 3$ mi/hr

Want $\frac{dA}{dt}$ when $y = 6$

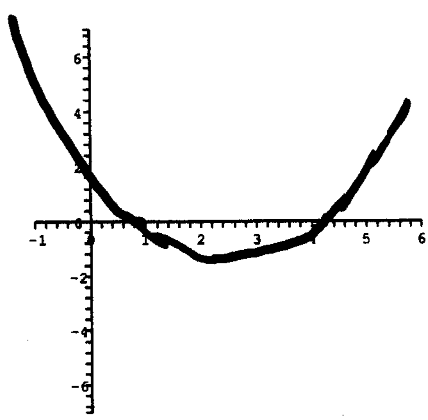
When $y = 6$, then $x = 10$

$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{dx}{dt}\cdot y + x\cdot\frac{dy}{dt}\right)$$
$$= \frac{1}{2}(5\cdot 6 + 3\cdot 10)$$
$$= 30 \text{ mi}^2/\text{hr}$$

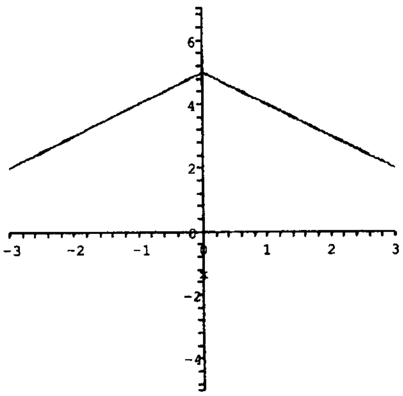
9. Consider the following graph of a function $f(x)$.



On the axes below, sketch a graph the derivative of $f(x)$ without "guessing" what the function is.



10. Consider the function $f(x) = 5 - |x|$, with graph given by



Is $f(x)$ differentiable at 0? Using the definition of derivative, explain why or why not.

No, it's not.

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{5 - |h| - 5}{h} = \lim_{h\to 0} \frac{-|h|}{h}$$

Since $\lim_{h\to 0^+} \frac{-|h|}{h} = -1$

and $\lim_{h\to 0^-} \frac{-|h|}{h} = +1$,

the limit doesn't exist, so the derivative doesn't exist at 0.